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# Cospectral graphs of the Grassmann graphs (Algebraic Combinatorics)

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## Cospectral graphs of the Grassmann graphs

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(joint work with Edwin van Dam)

Let  $q$  be a prime power, and  $V$  be a  $n$ -dimensional space over the  $GF(q)$  the field with  $q$  elements. Let  $1 \leq e \leq n - 1$  be an integer.

The **Grassmann Graph**  $G_q(n, e)$  has as vertices the  $e$ -dimensional subspaces and  $S \sim T$  iff their intersection is  $(e - 1)$ -dimensional.

To construct graphs with the same spectrum as  $G_q(n, e)$  we first will look at a partial linear space.

Let  $n, e$  be positive integers such that  $4 \leq 2e \leq n$ .

Let  $V$  be a  $n$ -dimensional vector space over  $GF(q)$  and

let  $H$  be a  $2e$ -dimensional subspace of  $V$ .

We first construct the partial linear space

$$\mathcal{LG}_q(n, e, e + 1).$$

Its points are the  $e$ -dimensional subspaces of  $V$ .

There are two kinds of lines:

Lines of the first kind:  $(e + 1)$ -dimensional subspaces  $L$  of  $V$  which are not a subspace of  $H$ . A line  $L$  has as points the  $e$ -dimensional subspaces contained in  $L$ .

b Lines of the second kind:  $(e - 1)$ -dimensional spaces  $M$  contained in  $H$ . A line  $M$  has as points the  $e$ -dimensional spaces contained in  $H$  which contain  $M$  as a subspace.

Now  $\mathcal{LG}_q(n, e, e + 1)$  has

$\begin{bmatrix} n \\ e \end{bmatrix}$  points,

$\begin{bmatrix} n \\ e+1 \end{bmatrix}$  lines,

each point is incident with  $\begin{bmatrix} n-e \\ 1 \end{bmatrix}$  lines

and each line is incident with  $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$  points.

Through any pair of points there is at most one line.

If  $P$  and  $Q$  are points then they are on a line iff  $P \cap Q$  is  $(e - 1)$ -dimensional.

Define  $P_q(n, e + 1)$  as the line graph of  $\mathcal{LG}_q(n, e, e + 1)$ , that is its vertices are the lines of  $\mathcal{LG}_q(n, e, e + 1)$  and two lines are adjacent iff they have exactly one point in common.

**Theorem 1** (i)  $P_q(n, e + 1)$  is cospectral with  $G_q(n, e + 1)$ ,  
(ii)  $P_q(n, e + 1)$  is distance-regular iff  $n = 2e + 1$ .  
(iii)  $P_q(2e + 1, e + 1)$  is not isomorphic to the Grassmann graph  $G_q(2e + 1, e + 1)$ .

(i) Let  $N$  be the point-line incidence matrix. Then  $NN^T - \begin{bmatrix} n-e \\ 1 \end{bmatrix}I$  is the adjacency matrix of the point graph. As the point graph is clearly  $G_q(n, e)$ , we know the spectrum of  $NN^T$ . Now except for the zero eigenvalue the spectrum of  $NN^T$  is the same as the  $N^TN$ . This implies that  $P_q(n, e+1)$  is cospectral with  $G_q(n, e+1)$  as  $NN^T - \begin{bmatrix} e+1 \\ 1 \end{bmatrix}I$  is the adjacency matrix for  $P_q(n, e+1)$ .

(ii) If  $n < 2e + 1$ , then there is  $e + 1$ -dimensional space  $L$  which intersects  $H$  in a  $(e - 1)$ -dimensional space  $M$ . Now in  $P_q(n, e + 1)$  the distance between  $L$  and  $M$  is 2 and it is easy to see that they have  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} e+1 \\ 1 \end{bmatrix}$  common neighbours where in the Grassmann graph  $c_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2$ .

If  $n = 2e + 1$ , then it is possible to check that it is distance-regular. An easy way to see this is true we use a result by Fiol and Garriga which states that if a graph has the same spectrum as a distance-regular graph  $\Gamma$  with diameter  $d$  is distance-regular iff for all vertices  $x$  we have  $k_d(x) = k_d(\Gamma)$ . And this is easily checked.

(iii) Let  $n = 2e + 1$ . Let  $K$  be an  $(e + 2)$ -dimensional space which intersects  $H$  in  $e + 1$  dimensions. Now the  $(e + 1)$ -dimensional subspaces of  $K$  which are not contained in  $H$  form a maximal clique of size  $\begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$  in  $P_q(2e + 1, e + 1)$ , whereas the Grassmann graph  $G_q(2e + 1, e + 1)$  has maximal cliques of sizes  $\begin{bmatrix} e+2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$ .

This shows the theorem.



(i) By looking at maximal cliques in  $P_q(2e + 1, e + 1)$ , it is easy to see that it is not vertex-transitive. The group  $P\Gamma L(2e + 1)_{2e}$  is an automorphism group of the graph. It was shown by M. Tagami that this is the full automorphism group.

(ii) For large  $q$  and  $e$  we were able to show that the local graph of a line of type 1 is not cospectral to the local graph of a line of type 2. We suspect that this is always the case. This implies, for example, that the Terwilliger Algebra depends on the base vertex for  $P_q(2e + 1, e + 1)$ .